

## Lyapunov exponents, noise-induced synchronization, and Parrondo's paradox

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We show that Lyapunov exponents of a stochastic system, when computed for a specific realization of the noise process, are related to conditional Lyapunov exponents in deterministic systems. We propose to use the term stochastically induced regularity instead of noise-induced synchronization and explain the reason why. The nature of stochastically induced regularity is discussed: in some instances, it is a dynamical analog of Parrondo's paradox.

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The apparent counterintuitive discovery that stochastic terms may increase coherence and induce order in a large variety of nonlinear systems has recently received considerable attention. Examples of such noise-induced order phenomena include stochastic resonance [1], coherence resonance [2], noise-induced synchronization [3,4], noise-induced pattern formation [5], spatiotemporal stochastic resonance [6], doubly stochastic resonance [7], to mention only a few. Noise-induced synchronization has been a rather controversial subject since its appearance [3,4]. In this paper (i) we explain that Lyapunov exponents of a stochastic system are related to conditional Lyapunov exponents introduced in the context of chaos synchronization by Pecora and Carroll [8]; (ii) we propose to use the term *stochastically induced regularity* instead of noise-induced synchronization and explain the reason why; and (iii) we discuss the nature of stochastically induced regularity: in some instances, it is a dynamical analog of Parrondo's paradox [9].

Many natural phenomena can be described as

$$\begin{aligned}\dot{\mathbf{u}} &= \mathbf{F}(\mathbf{u}), \\ \dot{\mathbf{v}} &= \mathbf{G}(\mathbf{v}, \mathbf{u}),\end{aligned}\tag{1}$$

where both  $\mathbf{u}$  and  $\mathbf{v}$  are assumed for simplicity to be  $d$ -dimensional vectors. We assume that Eq. (1) has an attractor  $\mathcal{A}$  and the driving system has an attractor  $\mathcal{A}_{dr}$ . We write  $\alpha_i$ ,  $i=1, \dots, d$ , for  $d$  Lyapunov exponents that correspond to the natural measure of the drive attractor. The system (1) has  $2d$  Lyapunov exponents  $\alpha_i$  and  $\lambda_i$ ;  $\lambda_i$  are often referred to as conditional Lyapunov exponents.

It might happen and we argue below that this in fact is very common that, when the drive is chaotic, the influence of this chaotic behavior to the response is effectively random. For example, consider the case for which the dynamics of the response is reduced to the following map:

$$x_{n+1} = f(x_n, \xi_n),\tag{2}$$

where  $f: M \rightarrow M$  is a (chaotic) mapping defined on the manifold  $M$ , and the second argument in  $f$  arises because  $f$  is

chosen randomly at each iterate  $n$  according to some rule. We view  $\xi_n$  as the limit of a deterministic process, e.g., we consider  $\xi_n = u_1(nT)$  to be the value of one of the coordinates of the trajectory  $\mathbf{u}(t)$ , say  $u_1$ , at time  $t = nT$ . If the dependence of  $\mathbf{u}(t)$  on  $t$  is chaotic and  $T$  is greater than some suitable correlation time of the flow, we may assume the variation of  $f$  with  $\xi_n$  to be effectively random and call Eq. (2) a random dynamical system [10]. With this point of view it makes sense to characterize the dynamics of Eq. (2) by Lyapunov exponents evolved under the same realization of the random process  $\xi_n$ , or equivalently, under the same initial condition of the flow dynamics  $\mathbf{u}$ . These exponents are exactly the conditional Lyapunov exponents. We stress that the largest Lyapunov exponent of Eq. (2) is *either not well-defined quantity or is infinity large number*. In contrast, the largest conditional Lyapunov exponent of Eq. (2) may be positive as well as negative. When the largest conditional Lyapunov exponent is negative for the stochastic system (2), we term this phenomenon as *stochastically induced regularity* in random dynamical systems. The case when the largest conditional Lyapunov exponent is negative for the deterministic system (1) is referred to as generalized synchronization [11].

One possible motivation for studying the stochastic models (2) comes from consideration of particles floating on the surface of a fluid whose flow velocity has a complicated time dependence [10]. Yu, Ott, and Chen has shown that in a particular example of Eq. (1), response can be reduced to a two-dimensional (2D) mapping known as Zaslavsky map [12] (when  $\xi_n = 0$ ), where  $\xi_n$  is independent random variables with uniform probability density in  $0 \leq \xi \leq 2\pi$ . Another motivation comes from biology. We may think that the drive term in Eq. (1) represents the effect of environmental fluctuation on the response, and model the response as additive and/or multiplicative stochastic process. In particular, we focus on *periodic dichotomous noise* process  $a(t)$  defined as follows: After lapses of fixed duration  $\tau$ ,  $a(t)$  takes the value  $a_0$  with probability  $p$  or  $a_1$  with probability  $1-p$ . Examples of such biologically motivated stochastic models can be found in Ref. [13].

We start our discussion with periodic dichotomous noise process; we consider the case, in which the dynamics of the response may be reduced to the following stochastic model:

$$\dot{\mathbf{v}} = \frac{1}{2}[1 - \nu(t)]\mathbf{G}_0(\mathbf{v}) + \frac{1}{2}[1 + \nu(t)]\mathbf{G}_1(\mathbf{v}),\tag{3}$$

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where  $\nu(t)$  is the continuous-time periodic dichotomous noise process

$$\nu(t) = \sum_{i=0}^{\infty} \xi_i \theta(t - i\tau) \theta((i+1)\tau - t), \quad t \geq 0,$$

driven by the discrete (Bernoulli trial) noise process  $\xi_i$  defined as  $\xi_i = 1$  with probability  $p$  and  $\xi_i = -1$  with probability  $1-p$ . In the last equation,  $\theta$  is the Heaviside function. The stochastic differential equation (3) can be integrated over the time interval  $\tau$  to give the two branches stochastic map, which can be rewritten as

$$\mathbf{x}_{n+1} = \mathbf{f}_i(\mathbf{x}_n), \quad (4)$$

where  $i=0,1$  and one of the maps  $\mathbf{f}_0, \mathbf{f}_1$  is applied randomly at each iteration with probability  $p$  and  $1-p$ , respectively. Stochastic processes (4) have been studied extensively in connection with stochastically induced coherence in bistable systems; a popular model of such process consists of two maps of the interval  $[0,1]$  given by  $f_i = cx + i(1-c)$ , where  $0 < c < 1$  is a parameter [14]. Unfortunately, it has been long known [15] that there are many values of  $c$  for which the invariant density of Eq. (4), with  $f_i = cx + i(1-c)$ , does not exist. However, in any experiment the dynamical state of the system cannot be precisely known and thus the stationary state can only be determined to a finite precision. A well-defined coarse-grained-invariant density *does* exist for the process (4) [16], and this is the quantity used here for all computations.

We now show that stochastically induced regularity can be observed in Eq. (4), when both  $\mathbf{f}_i$  are chaotic. The following example is a dynamical analog of Parrondo's paradox [9]. We start by considering two maps of the interval  $[0,1]$  defined as

$$f_0 = \begin{cases} \frac{x}{a}, & 0 < x < a, \\ \frac{ax}{1-a} - \frac{a^2}{1-a}, & a < x < 1, \end{cases} \quad (5)$$

and

$$f_1 = \begin{cases} \frac{ax}{1-a} + 1 - a, & 0 < x < 1 - a, \\ \frac{x}{a} - \frac{1-a}{a}, & 1 - a < x < 1, \end{cases} \quad (6)$$

where  $0 < a < 1$ . Both maps are chaotic with Lyapunov exponent:

$$\mu = \frac{1}{2-a} \ln\left(\frac{1}{a}\right) + \frac{1-a}{2-a} \ln\left(\frac{a}{1-a}\right).$$

In the last equation  $\mu$  is the (ordinary) Lyapunov exponent of the deterministic system  $x_{n+1} = f_i(x_n)$  where  $i$  is either 0 or 1. The stochastic process (4) has one conditional Lyapunov

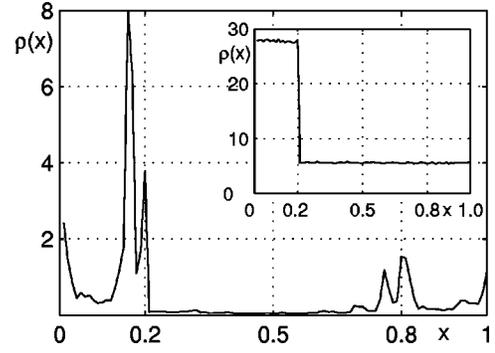


FIG. 1. Probability density of Eq. (4) with  $\mathbf{f}_i$  given by Eqs. (5) and (6) in the phase space  $[0,1]$  of the map  $f_0$ . The probability density in the phase space  $[0,1]$  of the map  $f_1$  is symmetric to the one shown here with respect to  $1/2$ . Inset figure: the probability density of the map  $f_0$  without switching.

exponents. Figure 1 shows the density of the natural (coarse-grained-invariant) measure for  $a=0.2$  and  $p=0.5$ . The corresponding conditional Lyapunov exponent is  $\lambda = -0.158$ . Figure 2 shows the conditional Lyapunov exponent of Eq. (4) versus the parameters  $a$  and  $p$ . We now explain why conditional Lyapunov exponent is negative. Let  $A=(0,a)$  and  $B=(a,1)$  be two subsets of the phase space of  $f_0$ . In a similar way, let  $C=(0,1-a)$  and  $D=(1-a,1)$  be two subsets of  $[0,1]$  for the map  $f_1$ . When only one map is applied, say  $f_0$ , the probability of a trajectory to visit the regions  $A$  and  $B$  is  $p_A = 1/(2-a)$  and  $p_B = (1-a)/(2-a)$ . Thus, for  $a=0.2$  we have  $p_A = 0.556$ ,  $p_B = 0.444$ , and  $\mu = 0.556 \ln 5 + 0.444 \ln 0.25 = 0.278$ . The probability of visiting the regions  $A, B, C$ , and  $D$  when the switching is allowed becomes, for  $a=0.2$  and  $p=0.5$ ,  $p_A = p_D = 0.205$ , and  $p_B = p_C = 0.295$ . Therefore,  $\lambda = 2 \times 0.205 \ln 5 + 2 \times 0.295 \ln 0.25 = -0.158$ . Without switching, a trajectory tends to spend a longer time in the left region  $A$  than in the right region  $B$  for the map  $f_0$ , and vice versa for the second map  $f_1$ : a trajectory tends to spend a longer time in the right region  $D$  than in the left region  $C$ . The coin flip erases this asymmetry. As a result of the flipping, the trajectory spends on average a

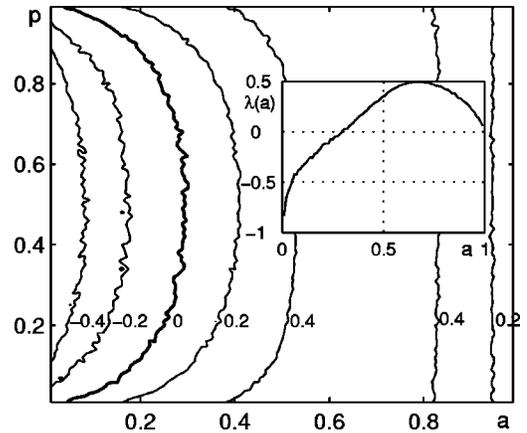


FIG. 2. Conditional Lyapunov exponent of Eq. (4) with  $\mathbf{f}_i$  given by Eqs. (5) and (6) versus the parameters  $a$  and  $p$ . Inset figure: conditional Lyapunov exponent  $\lambda(a)$  vs  $a$  for  $p=0.5$ .

longer time in the region with slope  $a/(1-a)$ , which results in negative conditional Lyapunov exponent.

The apparent paradox of the above example—that flip-flopping between two subsystems of a system, each of which independently has a property  $\mathcal{A}$  (for example, is chaotic), can allow a system to have a property  $\mathcal{B}$  (for example, to become nonchaotic) also applies to Brownian ratchets [20] and games of chance [21]. For instance, consider the game that illustrate Parrondo's paradox [21]. The game is played on  $1 \times 5$  checkerboard with a black square in the middle. The player moves the piece either forward or backward by rolling a pair of dice and consulting two rule sets. The object is to start in the middle and get to the right (winning) side before the left (losing) side. The first rule set is the piece moves forward from black if the sum of pair of rolling dice is 11, and from white if the sum is 7 or 11; the piece moves backward from black if the sum is 2,4 or 12, and from white if the sum is 2,3 or 12. The second rule set is identical to first, except for reversing the roles of black and white. In other words, the second rule set is the piece moves forward from white if the sum is 11, and from black if the sum is 7 or 11; the piece moves backward from white if the sum is 2,4 or 12, and from black if the sum is 2,3 or 12. If the player uses either set of rules, the player tends to lose. The relative probability of winning equals the number of ways to move forward from white to black times the number of ways to move forward from black to white, which is equal to  $8 \times 2$ . Losing involves moving backward twice, and the relative probability is equal to  $5 \times 4$ . Therefore, the player can expect to move forward only 80 times for every 100 backward moves.

However, randomly switching between the sets reverses the direction and the player tends to win. Now, the relative probability of winning is the average number of forward moves:  $[(8+2)/2][(8+2)/2]=25$ , while the relative probability of losing is  $[(4+5)/2][(4+5)/2]=20.25$ . Therefore, the player can expect to move forward 100 times for every 81 backward moves. The explanation is exactly the same as in our example with chaotic maps. Without switching, the piece tends to spend a longer time on a black square than on a white one and vice versa for the second set of rules. The coin flip erases this asymmetry. As a result of the flipping, two losing games become winning.

The phenomenon that has been just described is also observed for different nonlinear systems and/or stochastic processes. For example, instead of Eqs. (5) and (6) we use  $f_0 = 4x(1-x)$  and  $f_1 = 1 - 1/\sqrt{2} \sqrt{1 - (1 - |1 - 2x|)^2}$ , which are both chaotic and observe that periodic dichotomous noise with  $p=0.5$  results in a negative conditional Lyapunov exponent,  $\lambda = -0.11$  [17].

A deterministic system  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$  is called regular if its largest Lyapunov exponent is negative. In analog with this notion a stochastic process for which the largest conditional Lyapunov exponent is negative, is called *conditionally regular* stochastic process. For such system and for the same realization of the stochastic process, two different initial conditions will converge to the *same* (stochastic) trajectory. This explains the term *stochastically induced regularity* used

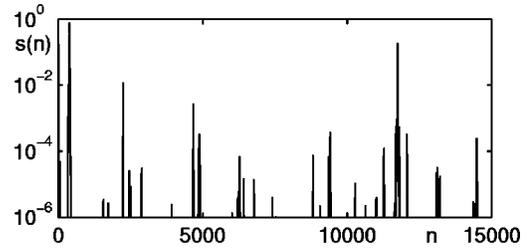


FIG. 3.  $s(n)$  vs  $n$  for Eq. (4) with  $\mathbf{f}_i$  given by Eqs. (5) and (6) and  $a=0.2$  and  $p=0.5$ .

through the paper. Therefore, in order to characterize stochastically induced regularity we define

$$s(n) = |x(n) - y(n)|,$$

where  $x(n)$  and  $y(n)$  are two trajectories of Eq. (4) evolved under the same realization of the random process  $\xi_n$ . When conditional Lyapunov exponent is negative,  $s(n)$  approaches zero as  $n$  goes to infinity. However, due to finite precision of computers,  $s(n)$  may become zero for finite time. To avoid this artificial result, small noise of the order of  $10^{-10}$  is added to obtain results in Fig. 3. Figure 3 shows  $s(n)$  as a function of time  $n$  for Eq. (4) with  $\mathbf{f}_i$  given by Eqs. (5) and (6) and  $a=0.2$  and  $p=0.5$ . We note that there are intervals of time where  $s$  is small, punctuated by the shorter intervals where  $s$  is of order 1. We can now offer a simple explanation of this behavior. An infinite random sequence of 0's and 1's contains arbitrary long but finite strings of only 0's (or 1's). If we interpret 0 and 1 as  $f_0$  and  $f_1$ , respectively, the last statement means that in a typical stochastic trajectory of Eq. (4) one can always find a finite sequence  $\{x_i\}_{i=l}^{i=l+k}$  obtained only through the iteration of the map  $f_0$ :  $x_{i+1} = f_0(x_i)$ . In such a case, since the map  $f_0$  is chaotic, two very close trajectories (of order  $10^{-10}$ ) will rapidly diverge from each other and, therefore,  $s$  will become of the order 1. Note that above conclusion holds for all  $a < a_c = 0.297$ , for which the conditional Lyapunov exponent is negative (Fig. 2). We have found (not reported here) the same intermittent behavior and explanation for different nonlinear systems and/or stochastic processes. Therefore, for conditionally regular stochastic processes  $s(n)$  *always approaches zero through the intermittent bursts* (in which  $s$  is of the order of the size of the attractor [18]), even when the largest conditional Lyapunov exponent is an arbitrary large negative number.

We now discuss the relation between stochastically induced regularity and synchronization. Two coupled identical chaotic systems, e.g.,  $x(t)$  and  $y(t)$ , exhibit synchronization of chaos if (i) the largest Lyapunov exponent along the manifold  $x=y$  is positive and (ii) the largest Lyapunov exponent normal to this manifold is negative. Both oscillators before and after the synchronization are chaotic, however, in the synchronous state  $x=y$ , which is different from the asynchronous state. In this light consider two 1D maps

$$\begin{aligned} x_{n+1} &= f(x_n) + \epsilon \xi_n, \\ y_{n+1} &= f(y_n) + \epsilon \xi_n, \end{aligned} \quad (7)$$

driven by a common stochastic function of time. The system (7) has two (conditional) Lyapunov exponents that are *always* equal to each other: both equations are exactly the same. Any “synchronization” in Eq. (7) is a consequence of the fact that the Lyapunov exponents associated with Eq. (7) are both negative, *two different initial conditions will converge to the same attractor*. It cannot happen that the Lyapunov exponent along the manifold  $x=y$  is positive and, at the same time, the Lyapunov exponent normal to this manifold is negative [19]. Therefore, stochastically induced regularity is a phenomenon that does not have any common properties with synchronization of periodic and/or chaotic signals. It may be also called suppression of chaos in sto-

chastic processes and is characterized by nonchaotic behavior of the deterministic part of the system (negative conditional Lyapunov exponents), while as a whole, the system is stochastic and, therefore, random and erratic.

To conclude, in this paper we have described a noise-induced order phenomenon that we term stochastically induced regularity. It is characterized with negative conditional Lyapunov exponents and intermittent behavior, it is related to Parrondo’s paradox but not to synchronization phenomena.

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